

Milne Universe, Tachyons, and Quantum Group

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We analyze the spectrum of the bosonic and superstring on the orbifold of the space-time by a boost, leading to the cosmological singularity. We show that the modular invariance leads to the spectrum where the twisted sector tachyon, together with all other twisted sector fields, present in the Euclidean version of the orbifold, is absent. This makes impossible to resolve the singularity by a marginal deformation of the worldsheet CFT. We also establish a relation between the resolution of rotational orbifolds in Euclidean and Lorentzian setups, and quantum groups. The analysis confirms the impossibility of resolving the cosmological orbifold singularity.

The article is based on two talks, given at “Workshop on ekpyrotic universe”, Annecy, 12/01, and at the XXX ITEP Winter school, Moscow, 02/01.

1. The model: Minkowskian orbifold

It was proposed in [1] to model the transition between a contracting and expanding phases of the Universe by an M-theory background of the form $\mathbf{R}^9 \times \mathcal{M}$, where \mathcal{M} is the so-called two dimensional Milne universe, the space with conical singularity:

$$ds^2 = -dt^2 + \lambda^2 t^2 dw^2 \tag{1.1}$$

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where w lives on a circle of the circumference 2π and $\lambda > 0$ is a parameter. If we interpret w as the eleventh dimensional circle, then from the ten dimensional perspective we have a Universe with the line element:

$$ds_{10}^2 = a^2(t) (-dt^2 + d\vec{x}^2), \quad a(t) \propto |t|^{\frac{1}{8}} \quad (1.2)$$

and the dilaton

$$\phi(t) = \phi(1) \pm \sqrt{\frac{2 \cdot 9}{8}} \log|t| \quad (1.3)$$

which passes through a zero string coupling point for the +1 choice of sign at $t = 0$. However the coupling is strong far away from the cosmic singularity. To study this configuration using string theory one is naturally led to compactifying one of the flat nine dimensions, then to do the nine-ten flip and interpret the same background as the strong coupling limit of the type IIA string propagating on the space-time of the form $\mathcal{M} \times \mathbf{R}^8$. The problem with the space \mathcal{M} is that the curvature has a delta-function singularity at $t = 0$. Therefore it fails to define a worldsheet conformal theory. Nevertheless, the manifold \mathcal{M} is an orbifold [2]. Take the Minkowski space $\mathbf{R}^{1,1}$, with the light-cone coordinates x^\pm :

$$x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^9) \quad (1.4)$$

The manifold \mathcal{M} is the quotient of the interior of the past- and future- light-cones $x^+x^- > 0$ by the action of the group \mathbf{Z} generated by a boost:

$$x^\pm \mapsto e^{\pm 2\pi\lambda} x^\pm \quad (1.5)$$

together with the point $x^\pm = 0$. It is not easy to confine the string to propagate in the interior of the light-cone, so instead we consider the orbifold of $\mathbf{R}^{1,1}$ by the same action \mathbf{Z} . The resulting space is non-Hausdorff (because of the light-cone) but the string propagation on it should be fine. Notice that the quotient M contains four cones, from the regions where x^\pm have some definite signs. In particular the domains with $x^+x^- < 0$ give rise to the space \mathcal{M}^\vee with the metric

$$ds^2 = dt^2 - \lambda^2 t^2 dw^2 \quad (1.6)$$

which contains closed time-like loops.

1.1. Particle in the orbifolded universe

Before studying the properties of a string, let us start with considering a relativistic particle of mass m propagating in $\mathbf{R}^{1,1}/\mathbf{Z}$. The wavefunction of this particle must be a \mathbf{Z} -invariant solution of Klein-Gordon equation on $\mathbf{R}^{1,1}$. Choose any on-shell momentum $p = (p^+, p^-)$ on the covering space: $2p^+p^- = m^2$. The function:

$$\Psi(x)_{p;l} = \int_{\mathbf{R}} dw e^{i(p^+x^-e^{-\lambda w} + p^-x^+e^{\lambda w} + lw)}, \quad l \in \mathbf{Z} \quad (1.7)$$

is clearly a \mathbf{Z} -invariant on-shell function. Moreover, changing (p^+, p^-) to $(z p^+, z^{-1} p^-)$, for $z > 0$, simply multiplies (1.7) by an irrelevant phase. By this transformation one can bring p to $\frac{1}{\sqrt{2}}(m, m)$ where m can be positive or negative. Let us introduce the appropriately normalized functions:

$$\Phi_{m,l}(x) = \sqrt{2|m|} \Psi_{\frac{1}{\sqrt{2}}(m,m);l}(x) \quad (1.8)$$

The functions (1.8) are orthonormal:

$$(\Phi_{m,l}, \Phi_{m',l'}) = \int_{\mathbf{R}^{1,1}/\mathbf{Z}} dx^+ dx^- \Phi_{m,l}^*(x) \Phi_{m',l'}(x) = \delta_{l,l'} \delta(m - m') \quad (1.9)$$

It is instructive to look at the behaviour of the wavefunctions $\Phi_{m,l}$ in the “bad” space-time region, i.e. for $x^+x^- < 0$. We claim that the motion of the massive particle in this region is finite. Indeed, for sufficiently large negative x^+x^- the function $\Phi_{m,l}(x)$ is exponentially decaying:

$$\Phi_{m,l} \sim \exp(-|m|\sqrt{-2x^+x^-}), \quad -x^+x^- \gg \frac{l^2}{2\lambda m^2} \quad (1.10)$$

1.2. Bosonic string

We proceed with consideration of the bosonic string propagating on $M \times \mathbf{R}^{24}$. The orbifold with respect to the group (1.5) contains twisted sectors, labeled by an integer M :

$$X^\pm(\sigma + \pi) = e^{\pm 2\pi M \lambda} X^\pm(\sigma) \quad (1.11)$$

We start with the $M = 1$ sector. The subsequent formulae are easily generalized to the general M case, by the substitution: $\lambda \rightarrow \lambda M$. The mode expansion of the X^\pm coordinates in the twisted sector is given by:

$$\begin{aligned} X_L^\pm &= \sum_n \frac{\tilde{\alpha}_n^\pm}{2(\pm\lambda - in)} e^{2(\pm\lambda - in)(t+\sigma)} \\ X_R^\pm &= - \sum_n \frac{\alpha_n^\pm}{2(\pm\lambda + in)} e^{-2(\pm\lambda + in)(t-\sigma)} \end{aligned} \quad (1.12)$$

Straightforward quantization leads to the commutation relations:

$$\begin{aligned} [\alpha_n^+, \alpha_m^-] &= \delta_{m+n}(m + i\lambda) \\ [\tilde{\alpha}_n^+, \tilde{\alpha}_m^-] &= \delta_{m+n}(m - i\lambda) \end{aligned} \quad (1.13)$$

Note that the oscillators with $n = m = 0$ form a Heisenberg algebra. The Virasoro generators L_n are calculated from the classical stress-energy tensor and then normal ordered with the result:

$$\begin{aligned} L_m &= - \sum_n \alpha_n^+ \alpha_{m-n}^- + \sum_n \alpha_n^i \alpha_{m-n}^i, \quad m \neq 0 \\ L_0 &= \frac{\lambda^2}{2} - \frac{1}{2} (\alpha_0^+ \alpha_0^- + \alpha_0^- \alpha_0^+) + \frac{1}{2} \vec{p}^2 + \sum_{n>0} \vec{\alpha}_{-n} \vec{\alpha}_n - \alpha_{-n}^+ \alpha_n^- - \alpha_{-n}^- \alpha_n^+ \end{aligned} \quad (1.14)$$

where $\vec{\alpha}_n$ are the transverse oscillators. The generators (1.14) form the standard algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{26}{12}(n^3 - n)\delta_{n+m}$$

The ground state is annihilated by the positive frequency oscillators:

$$\alpha_n^\mu |0\rangle_R = 0, \quad \tilde{\alpha}_n^\mu |0\rangle_L = 0, \quad n > 0 \quad (1.15)$$

The zero modes α_0^\pm have to be quantized with some care.

1.3. One-loop partition function

In this section we shall find that the correct way to quantize the zero modes α_0^\pm is by analytic continuation of the ordinary creation-annihilation operators of the harmonic oscillator to the case of imaginary frequency (just as in the open string case [3]).

Consider the one-loop string measure. It splits as a sum over all winding sectors, which we shall label by two integers (L_1, L_2) . In the (L_1, L_2) sector the boundary conditions on the worldsheet field X^\pm are:

$$X^\pm(z + m + n\tau) = \exp \pm 2\pi\lambda (L_1 m + L_2 n) X^\pm(z) \quad (1.16)$$

where we work on the Euclidean torus with the complex structure specified by the modular parameter $\tau = \tau_1 + i\tau_2$. We might also have to use the Lorentzian tori, for which τ and $\bar{\tau}$ are both real and $\tau - \bar{\tau} > 0$. In each case the metric on the torus is given by:

$$ds^2 = (dx + \tau dy)(dx + \bar{\tau} dy) \quad (1.17)$$

where $(x, y) \sim (x + m, y + n)$, $m, n \in \mathbf{Z}$. There are several ways of computing $\mathcal{Z}_{(L_1, L_2)}(\tau)$. One can directly calculate the regularized determinant of the Laplace operator, or one can calculate $\mathcal{Z}_{(0, L)}$ and then use modular group to generate the rest. The full partition function

$$\mathcal{Z}(\tau, \bar{\tau}) = \sum_{(L_1, L_2) \in \mathbf{Z}^2} \mathcal{Z}_{(L_1, L_2)}(\tau) \quad (1.18)$$

should be modular invariant. It is clear that (L_1, L_2) transforms as a doublet of $SL_2(\mathbf{Z})$. We expect that

$$\mathcal{Z}_{(aL_1 - cL_2, -bL_1 + dL_2)}\left(\frac{a\tau + b}{c\tau + d}\right) = \mathcal{Z}_{(L_1, L_2)}(\tau) \quad (1.19)$$

for $ad - bc = 1$. We can split the sum over (L_1, L_2) as a sum over the $N = \gcd(L_1, L_2)$ and then for fixed $N \neq 0$ the sum over the mutually prime integers (p, q) , s.t. $L_1 = Np, L_2 = Nq$. The latter sum is the sum over the coset space $SL_2(\mathbf{Z})/\mathbf{Z}$. In the subsequent integration over the modular domain

$$\int_{\mathcal{M}_{1,1}} \frac{d\tau d\bar{\tau}}{\tau_2^2} \mathcal{Z}(\tau, \bar{\tau}) \quad (1.20)$$

this summation unfolds the moduli space $\mathcal{M}_{1,1}$ and makes it into a strip $|\tau_1| \leq \frac{1}{2}, \tau_2 > 0$.

For $N = 0$ the boundary conditions are periodic and one gets the same answer as in the ordinary bosonic string case:

$$\mathcal{Z}_{(0,0)}(\tau) = \text{Vol}(\mathcal{M} \times \mathbf{R}^{24}) \frac{1}{\tau_2^{12} |\eta(\tau)|^{48}} \quad (1.21)$$

with the only difference being the reduction of the volume of the target space. For $N \neq 0$ the partition function may or may not contain the divergent factor coming from the zero modes, which are always present for $N = 0$. The 24 transverse coordinates will of course contribute the factor of $\text{Vol}(\mathbf{R}^{24})$ independently of N, λ .

Let us now analyze the conditions on the zero modes coming from the $\mathbf{R}^{1,1}/\mathbf{Z}$ part. The eigenvalues of the Laplace operator on the functions obeying (1.16), for the metric (1.17), are:

$$\lambda_{m,n} = - \left(\frac{2\pi}{\tau_2} \right)^2 (w + m + n\tau)(-\bar{w} + m + n\bar{\tau}), \quad w = i\lambda(L_1\tau - L_2) \quad (1.22)$$

It is clear that the modular group $SL_2(\mathbf{Z})$ permutes the eigenvalues $\lambda_{m,n}$ belonging to the different winding sectors. Now, we see that if the modular parameter τ is an $SL_2(\mathbf{Z})$ transform of $i\lambda \frac{p}{q}$, for $p, q \in \mathbf{Z}_+$,

$$\tau = \frac{a\tau_0 + b}{c\tau_0 + d}$$

then among the eigenvalues $\lambda_{m,n}$ there will be a zero, for the winding sector $(L_1, L_2) = (ap, cp)$. Conversely, for fixed $N = \gcd(L_1, L_2)$ there will be a finite number of points τ_0 in the fundamental domain $|\tau| \geq 1$, $|\tau_1| \leq \frac{1}{2}$ of $SL_2(\mathbf{Z})$ such that the Laplacian will have a zero mode obeying (1.16)²

If τ is generic then the spectrum of the Laplacian does not contain zero. The regularized determinant is given by the standard formula:

$$\text{Det}(-\Delta) = \exp - \zeta'(0), \quad \zeta(s) = \sum_{m,n} \frac{1}{\lambda_{m,n}^s} \quad (1.23)$$

It is easily computed to be

$$\text{Det}(-\Delta)^{-1} = \left| \frac{\eta(\tau)}{\vartheta_1(w|\tau)} \right|^2 \exp \left(\frac{2\pi w_1^2}{\tau_2} \right) \quad (1.24)$$

Full partition function We can now assemble all the pieces to write down the partition function for fixed N :

$$\mathcal{Z}_N(\tau, \bar{\tau}) = \sum_{\gcd(p,q)=1} \frac{d\tau d\bar{\tau}}{\tau_2^{13} |\eta(\tau)|^{42}} e^{G(Nw_{p,q}|\tau)} \quad (1.25)$$

where

$$G(z, \tau) = -\log \left| \frac{\vartheta_1(z|\tau)}{\vartheta_1'(0|\tau)} \right|^2 + 2\pi \frac{z_1^2}{\tau_2} \quad (1.26)$$

is the bosonic Green's function on the torus.

CFT interpretation Now let us go back to the $\mathbf{R}^{1,1}/\mathbf{Z}$ contribution, of a fixed (M, L) winding sector:

$$Z_{M,L}(\tau, \bar{\tau}) = \left| \frac{\eta(\tau)}{\vartheta_1(i\lambda(M\tau - L)|\tau)} \right|^2 e^{-2\pi\lambda^2 M^2 \tau_2} \quad (1.27)$$

It is easy to check the following modular properties of (1.27):

$$\begin{aligned} Z_{M,L} \left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}} \right) &= Z_{L,-M}(\tau, \bar{\tau}) \\ Z_{M,L}(\tau + 1, \bar{\tau} + 1) &= Z_{M,L-M}(\tau, \bar{\tau}) \end{aligned} \quad (1.28)$$

² Notice that for the Euclidean orbifold, by the rotation $X^+ = \overline{X^-} \mapsto e^{2\pi i \lambda} X^+$, the analogous phenomenon occurs only for rational λ , where for N which is divisible by the denominator of λ the Laplacian will have zero modes. However in that case (L_1, L_2) should be defined modulo this denominator as the twisting phases entering (1.16) are periodic in L_1, L_2 .

We would like now to perform a check of our analysis of the old covariant quantization. Consider the (pre)Hilbert space \mathcal{H}_M obtained by quantization in the M -th winding sector. Let g be the generator of the orbifold group \mathbf{Z} acting in \mathcal{H}_M . We expect:

$$\mathrm{Tr}_{\mathcal{H}_M} g^L q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \stackrel{?}{=} Z_{M,L}(\tau, \bar{\tau}) \quad (1.29)$$

Let us represent the Hilbert space as a tensor product of the left and right-moving spaces. We see from (1.27) that the spectrum of L_0 and \bar{L}_0 is given by:

$$\begin{aligned} L_0 &= \frac{(\lambda M)^2}{2} + (N_0 + \frac{1}{2})i\lambda M + \sum_{n>0} N_n^+(n + i\lambda M) + N_n^-(n - i\lambda M) \\ \bar{L}_0 &= \frac{(\lambda M)^2}{2} + (\bar{N}_0 + \frac{1}{2})i\lambda M + \sum_{n>0} \bar{N}_n^+(n - i\lambda M) + \bar{N}_n^-(n + i\lambda M) \end{aligned} \quad (1.30)$$

The range of N_0 and \bar{N}_0 is slightly ambiguous: one can take $N_0 \geq 0$ or $N_0 < 0$ and similarly for \bar{N}_0 . It actually depends on L what branch to choose (it has to do with the spectral flow).

1.4. Absence of the tachyon

The major consequence of this analysis is that the twisted sector contains no physical states. Indeed, the spectrum of L_0 and \bar{L}_0 is never real (the imaginary part is λ times half-integer), therefore the Virasoro constraints $L_0 = \bar{L}_0 = 1$ will have no solutions. In the analogous case of Euclidean rotational orbifold $\mathbf{R}^2/e^{2\pi i\lambda}$, the Virasoro generators differ from (1.14) by $\lambda \rightarrow i\lambda$ and one finds a physical spectrum, which turns out to contain a tachyon (even in the case of superstring).

This conclusion about the absence of the physical states remains correct in the case of superstring. Indeed, consider the NSR string on this orbifold background and perform the old covariant quantization [4]. The tachyon can only come from the NS sector. The twisted sector now in addition to the bosonic fields obeying (1.11) contains the left and right-moving fermions $\psi_{L,R}$ obeying:

$$\psi_{L,R}^\pm(\sigma + \pi) = -e^{\pm 2\pi M\lambda} \psi_{L,R}^\pm(\sigma) \quad (1.31)$$

which amounts to the mode expansion (we set $M = 1$ for simplicity):

$$\begin{aligned} \psi_L^\pm &= \frac{1}{2} \sum_{r \in \mathbf{Z} + \frac{1}{2}} \tilde{b}_r^\pm e^{2(\pm\lambda - ir)(\tau + \sigma)} \\ \psi_R^\pm &= \frac{1}{2} \sum_{r \in \mathbf{Z} + \frac{1}{2}} b_r^\pm e^{-2(\pm\lambda + ir)(\tau - \sigma)} \end{aligned} \quad (1.32)$$

The components b_r, \tilde{b}_r have the following reality constraints:

$$(b_r^\pm)^\dagger = b_{-r}^\pm, \quad (\tilde{b}_r^\pm)^\dagger = \tilde{b}_{-r}^\pm \quad (1.33)$$

and the canonical anti-commutation relations:

$$\{b_r^+, b_s^-\} = -\delta_{r+s}, \quad \{\tilde{b}_r^+, \tilde{b}_s^-\} = -\delta_{r+s} \quad (1.34)$$

The λ^2 shift in the L_0 generator of Virasoro algebra now cancels between bosons and fermions. However the imaginary part of L_0 is not cancelled:

$$L_0 = L_0^{bos} - \frac{\lambda^2}{2} + \sum_{r \geq \frac{1}{2}} -(r + i\lambda)b_{-r}^- b_r^+ - (-r + i\lambda)b_{-r}^+ b_r^- + r \vec{b}_{-r} \cdot \vec{b}_r \quad (1.35)$$

One – loop superstring measure The type IIA GSO projected one-loop measure is given by (after unfolding):

$$A = \sum_{L=1}^{\infty} Z(i\lambda L)$$

$$Z(u) = \int \frac{d^2\tau}{\tau_2^5} \frac{|\vartheta_{00}(u)\vartheta_{00}^3(0) - \vartheta_{01}(u)\vartheta_{01}^3(0) - \vartheta_{10}(u)\vartheta_{10}^3(0)|^2}{|\vartheta_{11}(u)\eta^9(\tau)|^2} = \quad (1.36)$$

$$\int \frac{d^2\tau}{\tau_2^5} \left| \frac{\vartheta_{11}^4(u/2)}{\vartheta_{11}(u)\vartheta_{11}^3(0)} \right|^2$$

It has poles at $\tau_{w,r,L} = \frac{i\lambda L - r}{w}$ where either w or r is not even (otherwise the pole from the denominator is cancelled by the zero of the numerator). To derive (1.36) it is recommended to consult p. 33, vol. 2 of [5], and to use some Riemann identities.

1.5. Stability

It seems that the absence of the tachyon implies the stability of the background. However one should keep in mind the example of the open string in the electric field, where the absence of physical states coexists with Schwinger instability having to do with the creation of the charged open strings. The signal of this instability – the poles in the one-loop measure — is currently under investigation [6].

2. Orbifolds and quantum group

In this section we discuss probing the orbifold geometry by D-branes. This topic is rather well-studied, however we believe that our observation is new. We shall consider specifically two-dimensional orbifolds.

First, let us look at the Euclidean orbifold: \mathbf{R}^2/Γ where Γ is a discrete subgroup of the rotation group $SO(2)$. We shall assume that Γ is generated by the rotation by an angle $2\pi\lambda$. If λ is rational then the group Γ is finite and the quotient is a cone. Let $q = e^{2\pi i\lambda}$.

Now consider placing a D-particle on this quotient (times otherwise flat space-time $\mathbf{R}^{1,7}$). As usual, it lifts to a Γ -orbit on the covering space. The gauge theory supported by the D-particle worldvolume is a truncation of the maximally supersymmetric $U(|\Gamma|)$. We should be thinking of $\mathcal{H} = \mathbf{C}^\Gamma$ – the space of functions on the orbit, as of the Chan-Paton space. The operation of permuting the pre-images of the D-particle according to the Γ -action leads to the unitary representation of Γ in \mathcal{H} which is the regular representation. We shall consider only regular branes. For these branes the truncation is simple. Namely, the adjoint scalars Z, Z^* corresponding to the motion along the two-dimensional space we are taking quotient of, and the adjoints X^i corresponding to the rest of the space, must obey:

$$\begin{aligned} U^{-1}ZU &= qZ \\ U^{-1}Z^*U &= q^{-1}Z^* \\ U^{-1}X^iU &= X^i \end{aligned} \tag{2.1}$$

where U is some unitary operator, representing the action of the rotation by $2\pi\lambda$ in the Chan-Paton space. These conditions imply that the spectrum of Z, Z^* is Γ -invariant. The gauge group is the subgroup of $U(|\Gamma|)$ which commutes with U .

The gauge theory action contains a potential $[Z, Z^*]^2$. In the vacuum $[Z, Z^*] = 0$ (technically speaking we should be taking Dp-branes with $p > 1$ to speak about well-localized vacua — we shall keep this point in mind but will be speaking about D-particles for simplicity). Thus, the vacua are the representations of the following algebra:

$$\begin{aligned} U^{-1}ZU &= qZ \\ U^{-1}Z^*U &= q^{-1}Z^* \\ [Z, Z^*] &= 0 \end{aligned} \tag{2.2}$$

It is obvious that the irreducible representations are labelled by the points on the quotient space. Indeed, as Z and Z^* commute, they can be diagonalized simultaneously. Let z be

the eigenvalue of Z . Then qz, q^2z, \dots are also eigen-values of Z . Hence in the irreducible representation the spectrum of Z coincides with an orbit of Γ . Moreover, replacing z by $q^l z$ can be undone by a gauge transformation with U^l . Hence the irreps up to an isomorphism are labelled by the orbits.

We can even calculate the metric on the quotient space in terms of the representation theory of the algebra (2.2). Let us write down the kinetic term of the D-particle Lagrangian:

$$\text{Tr} \nabla Z \nabla Z^*,$$

where $\nabla Z = \dot{Z} + [A_0, Z]$, and integrate out A_0 . We shall get the kinetic term

$$g_{z\bar{z}} \dot{z} \dot{\bar{z}}$$

where (z, \bar{z}) are some coordinates on the moduli space of irreps of (2.2), i.e. on the quotient space, and $g_{z\bar{z}}$ is the metric on the quotient.

Now let us give vev to some of the twisted sector fields. As we discussed in the previous section there are physical states in the twisted sector. As was shown in [7] the leading effect (coming from the disk amplitude with one insertion of the twisted sector field at the center and two open string vertex operators at the boundary) of the NSNS sector twisted states is to replace the potential $[Z, Z^*]^2$ by $([Z, Z^*] - \sum_l \mu_l U^l)^2$. Here μ_l is the vev of the tachyon in the l 'th twisted sector (corresponding to the rotation by $2\pi l\lambda$). In this sum one can in principle add the $l = 0$ term. This term corresponds to turning on the constant B -field along the Z plane, and does not come from the twisted sector. For simplicity we shall assume that $\mu_0 = 0$. The vacua of the D-particle gauge theory now get deformed to:

$$\begin{aligned} U^{-1} Z U &= q Z \\ U^{-1} Z^* U &= q^{-1} Z^* \\ [Z, Z^*] &= \sum_l \mu_l U^l \end{aligned} \tag{2.3}$$

Let us denote the algebra defined by (2.3) by $\mathcal{A}_{q,\mu}$. The irreps of (2.3) are still labelled by points of a two-dimensional manifold. And the same kinetic term technique gives a metric on this manifold. This metric now depends on μ and is generically smooth. Here is a quick way of calculating it. Let us work in the basis in \mathcal{H} where U is diagonal. Let $e^{i\theta}$ be the eigenvalue of U . Of course, it is easy to show that $\theta = 2\pi\lambda l$, $l \in \mathbf{Z}$ but we shall not use this. It follows from (2.3) that

$$Z = e^{\pi\lambda\partial_\theta} z(\theta) e^{\pi\lambda\partial_\theta}, \quad Z^* = e^{-\pi\lambda\partial_\theta} \bar{z}(\theta) e^{-\pi\lambda\partial_\theta} \tag{2.4}$$

and that:

$$z(\theta + \pi\lambda)\bar{z}(\theta + \pi\lambda) - z(\theta - \pi\lambda)\bar{z}(\theta - \pi\lambda) = \sum_l \mu_l e^{il\theta} \quad (2.5)$$

The product $g(\theta) = z(\theta)\bar{z}(\theta)$ is gauge invariant. From (2.5) we find:

$$g(\theta) = g + r(\theta), \quad r(\theta) = \sum_l \frac{\mu_l}{q^{l/2} - q^{-l/2}} e^{il\theta}$$

By the gauge transformation commuting with U we can bring Z, Z^* to the form (2.4) with $z(\theta) = \sqrt{g(\theta)}e^{i\varphi}$, $\bar{z}(\theta) = \sqrt{g(\theta)}e^{-i\varphi}$. Then the metric on the moduli space, parametrized by g, φ is calculated to be

$$\frac{dg^2}{4g_{eff}} + g_{eff}d\varphi^2, \quad \frac{1}{g_{eff}} = \sum_\theta \frac{1}{g(\theta)} \quad (2.6)$$

If $\lambda = \frac{1}{N}$ then $\theta = \frac{k}{N}$, $k = 0, \dots, N-1$. At the orbifold point, $\mu_l = 0$, $g_{eff} = \frac{g}{N}$, and the metric (2.6) has \mathbf{Z}_N singularity at $g = 0$. If μ_l are generic then $g_{eff} \rightarrow 0$ at $g \neq 0$, namely at $g = g_0 = -\min_\theta r(\theta)$, and locally the metric (2.6) will look like

$$\frac{dg^2}{4(g - g_0)} + (g - g_0)d\varphi^2 = dr^2 + r^2d\varphi^2$$

for $r^2 = g - g_0$. If there are several θ for which $r(\theta)$ reaches minimum then the singularity will be of $\mathbf{Z}_{N'}$ type where N' is the multiplicity of the minimum. The main conclusion here is that turning on μ 's one smoothes out the singularity. This conclusion is confirmed by other analysis [8][9].

Notice, that for a specific choice of μ one can get a famous algebra as the algebra of functions on the resolved quotient. Namely, let us take

$$\mu_l = \mu \frac{\delta_{l,1} - \delta_{l,-1}}{q - q^{-1}}, \quad \mu \in \mathbf{R}_+$$

Introduce an operator H , such that $U = q^H$, and the operators E and F , by the formulae: $E = Z/\sqrt{\mu}$, $F = Z^*/\sqrt{\mu}$. Then the commutation relations (2.3) become that of the quantum group $U_q sl_2$! Moreover the reality conditions are those of the quantum $SU_q(2)$. For finite Γ we need q to be the root of unity. This is the famous special case of the quantum group representation theory.

One can ask at this point, what about $SL(2)$? In that case the reality conditions would require E and F to be self-adjoint, and to have unitary representations one would need q to be real. But this is exactly the setup for the Lorentzian orbifold we started with!

Now, let us assume for a moment that the Lorentzian orbifold twisted sector had physical states³. Let us also assume for a moment that the slowly moving D-particle in the time-dependent background such as (1.1) is described by the Lorentzian version of the IKKT model. Then instead of static vacua we can talk about the solutions to the matrix equations of motion describing the (noncommutative) algebra of functions on the D-brane worldvolume (as opposed to the usually considered algebra of functions on its space-like section). We can now look at the Lorentzian version of (2.3) and investigate the metric on the moduli space.

One new feature compared to the $\Gamma = \mathbf{Z}_N$ case is that $\Gamma = \mathbf{Z}$ and the spectrum of θ is continuous, in fact $\theta \in \mathbf{R}/2\pi\mathbf{Z}$. The formula (2.6) for the metric is now changed to:

$$\frac{dg^2}{4g_{eff}} - g_{eff}d\varphi^2 \quad (2.7)$$

where now:

$$\frac{1}{g_{eff}} = \frac{1}{2\pi} \int d\theta \frac{1}{g(\theta)}, \quad g(\theta) = g + \sum_l \frac{m_l}{\sinh(\pi l \lambda)} e^{il\theta}$$

Now for g near the minimum of $r(\theta)$ the behaviour of g_{eff} is different:

$$g_{eff} \sim \sqrt{g - g_0}$$

and the metric (2.7) looks more singular than the original orbifold metric!

It would have been a very bizarre situation if by turning on the twisted sector fields one made the singularity worse than it was before! Luckily there are no twisted sector fields to turn on, and no contradiction.

Acknowledgements

I would like to thank G. Moore, N. Seiberg for their interest and discussions, C. Bachas for collaboration in [6], and A. Gorsky, I. Kogan, A. Losev, A. Morozov, and B. Pioline for useful discussions.

I would like to thank the organizers of the December 2001 workshop at Annecy and the organizers of ITEP winter school for warm hospitality and for invitation to present some of the results above. I am also grateful to the New High Energy Theory Center at

³ This would have been the case had the zero modes α_0^\pm been quantized in a unitary manner, i.e. $\alpha_0^- \sim x, \alpha_0^+ \sim i\lambda\partial_x$. In this case for the type IIA string one finds a physical spectrum in the sector with any winding number

Rutgers University for its kind invitation in October 2001 which helped me to perform this study. I am especially grateful to my family for its support and understanding.

Research was partially supported by RFFI under grant 00-02-16530 and under the grant 00-15-96557 for scientific schools, and by Clay Mathematical Institute.

Note added. After this paper has been completed we have received the manuscripts [10][11][12] which address somewhat related issues. In particular, the paper [10] also analyzes the bosonic string one-loop partition function in the Lorentzian orbifold background.

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